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Integral identities of potential theory of radiation and diffraction of regular water waves by a body

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SUMMARY

The study presents a new general integral identity for the velocity potential of flow about a body in regular water waves. This integral identity is valid outside, inside, and exactly on the surface of the body, and is equivalent to the set of three classical identities valid strictly outside, inside, and on the body, respectively. For the usual problem of wave radiation and diffraction by a body, the integral identity yields an integral equation for determining the potential on the body surface. An interesting feature of the integral identity and related integral equation obtained in this study is that they involve an integral of the Green function over the waterplane inside the body in the case of an open sea-surface piercing body. Alternatively, these equations can be expressed in terms of a modified Green function involving the previously-noted waterplane integral of the Green function.

1. Introduction

Diffraction and radiation of regular (time-harmonic) water waves by fixed structures (such as offshore terminals and drilling platforms) or moving bodies (moored tankers, floating offshore platforms, and wave-energy devices) is a subject of great practical importance in ocean engineering. Diffraction and radiation of regular waves by two-dimensional horizontal cylinders is also important in naval hydrodynamics for predicting ship motions within the framework of the slender-ship theory, as is explained in Newman [1] for instance.

With the advent of large computers, calculations of wave radiation and diffraction by three-dimensional bodies have become feasible. As a matter of fact, a number of computer programs, based on both integral-equation and finite-element methods, have been developed and used. A review of the development of these numerical methods may be found in Mei [2]; additional recent calculation methods have been developed by Guevel, Daubisse, and Delhommeau [3], and Martin [4]. However, calculations for arbitrary three-dimensional bodies are time consuming, and there is a need for seeking improved computational methods.

The present study is concerned with the method of integral equations. This classical method consists in formulating, and then solving numerically, an integral equation for the velocity potential or an assumed auxiliary distribution of sources or dipoles, as is explained for instance

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in John [5], Wehausen [6], and Mei [2]. Integral equations for source distributions were apparently first used by Kim [7] for elliptical cylinders and ellipsoids, by Daubert and Lebreton [8] and Frank [9] for general cylinders, and by Lebreton and Margnac [10] for three-dimensional bodies. Other applications of the integral-equation method are reviewed in [2]. This method is also used in [3].

The integral-equation method is based on the use of a Green function representing the velocity potential due to a pulsating source at a fixed position below the sea surface or a pulsating flux across the sea surface. The computing times involved in the integral-equation method directly depend on, indeed are essentially proportional to, the computing times required for evaluating the Green function. This motivated a search for single-integral representations and series representations of the Green function suited for efficient numerical evaluation. Several integral representations, obtained and used by various authors, are listed in Noblesse [11], where new series representations are also obtained. Specifically, an asymptotic expansion and a convergent ascending series are obtained; these series permit simple and efficient numerical evaluation of the Green function for large and small distances, respectively, from the mean-sea-surface mirror-image of the singularity (submerged source or flux across the sea surface). One-dimensional Taylor series expansions of the Green function in the neighborhood of the horizontal and vertical axes are also given in [11]. This study of the Green function is supplemented by the present study, concerned with the formulation of an integral equation for determining the velocity potential.

More precisely, a new basic integral identity satisfied by the potential is obtained. In the important particular case of radiation and diffraction of regular waves by a body, this general integral identity becomes an integral equation for calculating the potential on the surface of the body. Analogous integral identities and related integral equations have previously been obtained by the author for the problems of potential flow about a body in an unbounded fluid [12] and about a ship in steady rectilinear motion in a calm sea (the ship wave-resistance problem) [13]. The present study thus is closely related to, and indeed supplements, studies [11, 12, 13].

The plan and main results of this study are now presented. The basic potential-flow problem of the linearized theory of flow about a body in regular water waves is briefly formulated in Section 2. Also stated in this section are fundamental equations, specifically equations (2.4a, b) and equations (2.5a, b), satisfied by the Green function. Equations (2.4a, b) are well known. However, these equations are only valid strictly below the mean sea surface $\zeta = 0$, and equations (2.5a, b) are appropriate in the limit $\zeta = 0$, as was previously shown in [11].

The core of the study is Section 3, in which basic integral identities for the velocity potential are obtained. The three classical integral identities (3.6a, b, c) – valid strictly outside, inside, and on the surface of the body, respectively – are first obtained. However, the main result of Section 3 is the new integral identity (3.8), which is valid outside, inside, and exactly on the body surface, and in fact is equivalent to the set of the three usual identities (3.6a, b, c). An interesting feature of the integral identity (3.8) for an open sea-surface piercing body is the presence of the 'waterplane integral' w defined by equation (3.8a). This waterplane integral vanishes for a fully-submerged closed body, and integral identity (3.8) becomes identity (3.10). An alternative form of the integral identity (3.8) is given by identity (3.9) involving the 'modified Green function' $\tilde{G}$ defined by equation (3.9a). In the case of a fully-submerged closed body, the modified Green function $\tilde{G}$ becomes identical to the original Green function G, and identity (3.9) becomes identity (3.10).

For the important usual problem of wave radiation and diffraction by a body, the potential ϕ satisfies the Laplace equation $\nabla^2\phi = 0$ in the mean flow domain and the (linearized) sea-surface boundary condition $\partial\phi/\partial z - f\phi = 0$ on the mean sea surface, and the general integral identity (3.8) becomes equation (4.2), which is an integral equation for determining the potential on the surface b of the body. This integral equation is considered in Section 4.

2. The basic potential-flow problem and the Green function

The basic potential-flow problem of the linearized theory of flow about a body in regular water waves is briefly formulated below. A sea of large depth and horizontal extent is assumed. The only body force considered is that due to a uniform gravitational field, with acceleration g. Water is regarded as homogeneous and incompressible, with density ρ. Viscosity effects are ignored, and irrotational flow is assumed. Surface tension, wavebreaking, spray formation along the intersection between the surfaces of the body and the sea, and nonlinearities in the sea-surface condition are neglected. The mean (undisturbed) sea surface is taken as the plane $Z = 0$.

Only flows simple-harmonic in time, with radian frequency ω (period $2\pi/\omega$), are considered in this study. The velocity potential $\Phi'(\mathbf{X}, T)$ may thus be expressed in the form

$$\Phi'(\mathbf{X}, T) = \mathrm{Re}\, \Phi(\mathbf{X}) \exp(-i\omega T),$$

where the 'spatial component' $\Phi(\mathbf{X})$ of the actual potential $\Phi'(\mathbf{X}, T)$ only depends on the position vector $\mathbf{X}$ and not on the time T, and Re represents the real part of the complex function on the right side. Eventual (linearized) pressure distribution $P'(X, Y, T)$ and flux distribution $Q'(X, Y, T)$ at the mean sea plane likewise are assumed to be of the form

$$P'(X, Y, T) = \mathrm{Re}\, P(X, Y) \exp(-i\omega T),$$

$$Q'(X, Y, T) = \mathrm{Re}\, Q(X, Y) \exp(-i\omega T).$$

Nondimensional variables are defined in terms of $1/\omega$ as reference time and of some reference length L, from which the reference velocity ωL, potential ωL^2, and pressure $\rho\omega^2 L^2$ can be formed. The nondimensional variables

$$t = \omega T, \qquad \mathbf{x} = \mathbf{X}/L, \qquad \phi = \Phi/\omega L^2, \qquad p = P/\rho\omega^2 L^2, \qquad q = Q/\omega L$$

are then defined.

In terms of the above nondimensional variables, the basic potential-flow problem of the linearized theory of flow about a body in regular water waves can be stated as follows. The problem consists in solving the Laplace equation

$$\nabla^2\phi = 0 \quad \text{in} \quad d, \tag{2.1}$$

subject to boundary conditions specified below. The solution domain d in equation (2.1) is the domain exterior to the body and bounded upwards by the mean sea surface, σ say, which consists in the whole plane $z = 0$ if the body is fully submerged, or in the portion of the plane $z = 0$ exterior to the body if it pierces the sea surface.

On the body surface, b say, which actually consists in only the portion of the body surface located below the plane $z = 0$ if the body pierces the sea surface, the potential must satisfy the usual Neumann condition

$$\partial\phi/\partial n \quad \text{given on} \quad b, \tag{2.2}$$

where $\partial\phi/\partial n \equiv \nabla\phi \cdot \mathbf{n}$ is the derivative of ϕ in the direction of the unit normal vector $\mathbf{n}$ to b, taken to be pointing inside the fluid. The precise form taken by the expression for $\partial\phi/\partial n$ on b in particular problems, notably in the usual 'radiation' and 'diffraction' problems, may be found in various places in the literature, e.g., in Wehausen [6] and Newman [14].

On the mean sea surface σ, the potential must satisfy the (linearized) sea-surface condition

$$\partial\phi/\partial z - f\phi = ifp - q \quad \text{on} \quad \sigma, \tag{2.3}$$

where f is the nondimensional frequency parameter defined as

$$f = \omega^2 L/g. \tag{2.3a}$$

For most problems of practical interest, the pressure at the sea surface is constant equal to the atmospheric pressure, and there is no flux across the sea surface, so that one usually has $p = 0 = q$. A derivation of the sea-surface boundary condition (2.3) in the general case when sea-surface pressure and flux distributions are allowed may be found in [11]. Finally, the usual 'radiation condition,' expressing that waves at a sufficient distance away from the disturbance (here, a body) which created them are like 'outgoing' progressive waves, must be imposed for uniqueness.

The above-defined boundary-value problem can be solved by formulating an integral equation for the velocity potential based on the use of the Green function, denoted by $G(\boldsymbol{\xi}, \mathbf{x})$ or simply G, associated with the linearized sea-surface boundary condition (2.3). The Green function $G(\boldsymbol{\xi}, \mathbf{x})$ is the 'spatial component' of the velocity potential Re $G(\boldsymbol{\xi}, \mathbf{x}) \exp(-it)$ of the flow created at point $\boldsymbol{\xi}$ by a pulsating outflow of strenght Re exp(– it) located at point $\mathbf{x}$, stemming from a submerged point source if $z < 0$ or from a point flux across the mean sea surface if $z = 0$. Specifically, the Green function satisfies the following equations

$$\left.\begin{aligned} &\nabla^2 G = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta) \quad \text{in} \quad z<0 &(2.4a)\\ &\partial G/\partial z - fG = 0 \quad \text{on} \quad z = 0 &(2.4b)\end{aligned}\right\} \text{if} \quad \zeta < 0,$$

$$\left.\begin{aligned} &\nabla^2 G = 0 \quad \text{in} \quad z<0 &(2.5a)\\ &\partial G/\partial z - fG = -\delta(x-\xi)\delta(y-\eta) \quad \text{on} \quad z = 0 &(2.5b)\end{aligned}\right\} \text{if} \quad \zeta = 0.$$

Equations (2.4a, b) are well known. However, these equations are valid only for ζ strictly negative, and equations (2.5a, b) are proper in the limiting case $\zeta = 0$. A derivation of equations (2.4) and (2.5) is given in [11], where an ascending series and an asymptotic expansion for the Green function may also be found. Both equations (2.4) and (2.5) are used in the following section for obtaining integral identities for the velocity potential.

3. Fundamental integral identities

In this section, basic integral identities for the velocity potential are obtained by applying a classical Green identity to the potential $\phi \equiv \phi(\mathbf{x})$ and the previously-defined Green function $G \equiv G(\boldsymbol{\xi}, \mathbf{x})$. The case of a body piercing the sea surface is considered for definiteness. The Green identity is

$$\int_{d'} (\phi \nabla^2 G - G \nabla^2 \phi)\, dv = \int_{\sigma'} (\phi \partial G/\partial z - G \partial \phi/\partial z)\, dx dy$$

$$+ \int_b (G \partial\phi/\partial n - \phi \partial G/\partial n) da + \int_{b_\infty} (\phi \partial G/\partial n - G \partial \phi/\partial n) da, \tag{3.1}$$

where d' is the finite domain bounded by the body surface b, the mean sea plane $z = 0$, and some arbitrary, but sufficiently-large, exterior surface b_∞ surrounding the body surface b, as is shown in Figure 1; furthermore, σ' is the portion of the plane $z = 0$ between the intersection curves c and c_∞ of the plane $z = 0$ with the body surface b and the exterior surface b_∞, respectively. On the surfaces b and b_∞, we have $\partial\phi/\partial n \equiv \nabla\phi \cdot \mathbf{n}$ and $\partial G/\partial n \equiv \nabla G \cdot \mathbf{n}$ where $\mathbf{n}$ is the unit outward normal vector to b or b_∞, as is shown in Figure 1. Finally, dv and da represent the differential elements of volume and area at the integration point $\mathbf{x}$ of the domain d' and the surfaces b or b_∞, respectively, and $dxdy$ evidently is the differential element of area of the mean sea surface.

The integral over the exterior surface b_∞ in equation (3.1) can be shown to vanish as this large surrounding surface is made ever larger. This integral can then be ignored if the finite domain d' and the finite region σ' of the mean sea plane are replaced by the unbounded mean flow domain d and the unbounded mean sea surface σ outside the body surface b and its intersection curve c with the plane $z = 0$, respectively. By expressing the integrand $\phi\partial G/\partial z - G\partial\phi/\partial z$ of the sea-surface integral (3.1) in the form $\phi(\partial G/\partial z - fG) - G(\partial\phi/\partial z - f\phi)$, we may obtain

$$\int_d \phi \nabla^2 G\, dv - \int_\sigma \phi(\partial G/\partial z - fG)\, dx dy =$$

$$\int_d G \nabla^2 \phi\, dv - \int_\sigma G(\partial\phi/\partial z - f\phi)\, dx dy + \int_b (G \partial \phi/\partial n - \phi \partial G/\partial n)\, da. \tag{3.2}$$

By expressing the potential ϕ in the integrands of the two integrals on the left side of equation (3.2) in the form $\phi = \phi_* + (\phi - \phi_*)$, where $\phi \equiv \phi(\mathrm{x})$ as was defined previously, and ϕ_* represents the potential at the field point $\boldsymbol{\xi}$, that is $\phi_* \equiv \phi(\boldsymbol{\xi})$, we may obtain

$$\int_d \phi \nabla^2 G \, dv - \int_\sigma \phi(\partial G/\partial z - fG) dx dy = C\phi_* + C', \tag{3.3}$$

where C and C' are defined as

$$C = \int_d \nabla^2 G dv - \int_\sigma (\partial G/\partial z - fG) dx dy, \tag{3.4}$$

$$C' = \int_d (\phi - \phi_*) \nabla^2 G dv - \int_\sigma (\phi - \phi_*)(\partial G/\partial z - fG) dx dy.$$

It may be seen from equations (2.4) and (2.5) that we have $C' \equiv 0$ if $\phi - \phi_* \equiv \phi(\mathrm{x}) - \phi(\boldsymbol{\xi}) \to 0$ as $\mathrm{x} \to \boldsymbol{\xi}$, that is if the potential is continuous everywhere in the solution domain d and on its boundary $\sigma + b + c$, as is assumed here. Use of equation (3.3), with $C' = 0$, in equation (3.2) then yields

$$C\phi_* = \int_d G\nabla^2 \phi dv - \int_\sigma G(\partial\phi/\partial z - f\phi) dx dy + \int_b (G\partial\phi/\partial n - \phi\partial G/\partial n) da, \tag{3.5}$$

where C is given by formula (3.4).

Use of equations (2.4) and (2.5) in expression (3.4) for C shows that we have $C \equiv 1$ if the field point $\boldsymbol{\xi}$ is strictly outside the body surface b, in d or on σ, whereas we have $C \equiv 0$ if $\boldsymbol{\xi}$ is strictly inside the body surface b. We thus have

$$\phi_* = \int_d G\nabla^2 \phi dv - \int_\sigma G(\partial\phi/\partial z - f\phi) dx dy + \int_b (G\partial\phi/\partial n - \phi\partial G/\partial n) da \tag{3.6a}$$

for $\boldsymbol{\xi}$ in the exterior domain $d + \sigma - b - \mathrm{c}$, and

$$0 = \int_d G\nabla^2 \phi dv - \int_\sigma G(\partial\phi/\partial z - f\phi) dx dy + \int_b (G\partial\phi/\partial n - \phi\partial G/\partial n) da \tag{3.6b}$$

for $\boldsymbol{\xi}$ in the interior domain $d_i + \sigma_i - b - c$ where d_i and σ_i represent the domain and the portion of the plane $z = 0$, respectively, strictly inside the body surface b, as is shown in Figure 1, so that $d + d_i + b$ and $\sigma + \sigma_i + c$ are the whole lower half space $z < 0$ and the whole plane $z = 0$, respectively. It can also be seen from equations (2.4) and (2.5) that we have $C = 1/2$ if the point $\boldsymbol{\xi}$ is exactly on the body surface b or its intersection c with the plane $z = 0$, at least for points $\boldsymbol{\xi}$ where $b + c$ is smooth; more generally, the value of $4\pi C$ (or $2\pi C$)

at a point $\boldsymbol{\xi}$ of b (or c) is equal to the angle at which d(or σ) is viewed from the point $\boldsymbol{\xi}$. We then have

$$\phi_*/2 = \int_d G\nabla^2\phi \mathrm{d}v - \int_\sigma G(\partial\phi/\partial z - f\phi)\mathrm{d}x\mathrm{d}y + \int_b (G\partial\phi/\partial n - \phi\partial G/\partial n)\mathrm{d}a \tag{3.6c}$$

for $\boldsymbol{\xi}$ exactly on smooth $b + c$. Equations (3.6a, b, c) are well known, although the particular cases of these equations corresponding to $\nabla^2\phi = 0 = \partial\phi/\partial z - f\phi$ are usually given in the literature, and these equations are usually obtained from the Green identity (3.1) in a manner different from that shown here.

The value of the constant C on the left side of equation (3.5) is discontinuous across the body surface b, C being equal to 1 outside b and to 0 inside, as is explicitly indicated in equations (3.6). This discontinuity in the value of C evidently is accompanied by a corresponding discontinuity on the right side of equation (3.5). Specifically, the latter discontinuity stems from the integral $\int_b \phi\partial G/\partial n \ \mathrm{d}a$, which represents a potential associated with a distribution of normal dipoles on the body surface. An integral identity valid for any point $\boldsymbol{\xi}$ – outside, inside, or exactly on the body surface – can be obtained by eliminating the discontinuity in the value of C in equation (3.5). This can be done by adding the term $C_i\phi_*$ on both sides of equation (3.5), with C_i given by

$$C_i = \int_{d_i} \nabla^2 G \mathrm{d}v - \int_{\sigma_i} (\partial G/\partial z - fG)\mathrm{d}x\mathrm{d}y. \tag{3.7}$$

Use of the divergence theorem

$$\int_{d_i} \nabla^2 G \mathrm{d}v = \int_{\sigma_i} \partial G/\partial z \mathrm{d}x\mathrm{d}y + \int_b \partial G/\partial n \ \mathrm{d}a$$

in equation (3.7) yields the alternative expression

$$C_i = f\int_{\sigma_i} G\mathrm{d}x\mathrm{d}y + \int_b \partial G/\partial n \mathrm{d}a. \tag{3.7a}$$

By adding the term $C_i\phi_*$ on the left and right sides of equation (3.5), with C_i given by expression (3.7) on the left side and expression (3.7a) on the right side, we may obtain

$$\left(I - f\int_{\sigma_i} G\mathrm{d}x\mathrm{d}y\right)\phi_* = \int_d G\nabla^2\phi \mathrm{d}v - \int_\sigma G(\partial\phi/\partial z - f\phi)\mathrm{d}x\mathrm{d}y$$

$$+ \int_b [G\partial\phi/\partial n - (\phi - \phi_*)\partial G/\partial n] \ \mathrm{d}a,$$

where I is defined as

$$I = \int_{d+d_i} \nabla^2 G \, \mathrm{d}v - \int_{\sigma+\sigma_i} (\partial G/\partial z - fG)\,\mathrm{d}x\mathrm{d}y.$$

It can be seen from equations (2.4) and (2.5) that we have $I \equiv 1$ for any point $\boldsymbol{\xi}$ in the lower half space $\zeta \leqslant 0$.

We then have

$$(1 - fw_*)\phi_* = \int_d G\nabla^2\phi\,\mathrm{d}v - \int_\sigma G(\partial\phi/\partial z - f\phi)\,\mathrm{d}x\mathrm{d}y + \int_b [G\partial\phi/\partial n - (\phi - \phi_*)\partial G/\partial n]\,\mathrm{d}a, \tag{3.8}$$

where $w_* \equiv w(\boldsymbol{\xi})$ is the 'waterplane integral' given by

$$w(\boldsymbol{\xi}) = \int_{\sigma_i} G(\boldsymbol{\xi}, x, y, z = 0)\,\mathrm{d}x\mathrm{d}y. \tag{3.8a}$$

An alternative form of the integral identity (3.8) involving a 'modified Green function' $\tilde{G}$ can be obtained by dividing equation (3.8) by 1-fw_*. This yields

$$\phi_* = \int_d \tilde{G}\nabla^2\phi\,\mathrm{d}v - \int_\sigma \tilde{G}(\partial\phi/\partial z - f\phi)\,\mathrm{d}x\mathrm{d}y + \int_b [\tilde{G}\partial\phi/\partial n - (\phi - \phi_*)\partial\tilde{G}/\partial n]\,\mathrm{d}a, \tag{3.9}$$

where the 'modified Green function' $\tilde{G}$ is given by $\tilde{G} = G/(1 - fw_*)$, that is

$$\tilde{G}(\boldsymbol{\xi}, \mathbf{x}) = G(\boldsymbol{\xi}, \mathbf{x}) \Big/ \left[1 - f\int_{\sigma_i} G(\boldsymbol{\xi}, \mathbf{x})\,\mathrm{d}x\mathrm{d}y\right]. \tag{3.9a}$$

The above integral identities were derived for the case of an open body piercing the sea surface, as is shown in Figure 1. In the case of a closed body entirely submerged below the sea surface, we evidently have $\sigma_i \equiv 0$, so that the waterplane integral w vanishes and the modified Green function $\tilde{G}$ is identical to the original Green function G, and the alternative identities (3.8) and (3.9) take the same form, namely

$$\phi_* = \int_d G\nabla^2\phi\,\mathrm{d}v - \int_{z=0} G(\partial\phi/\partial z - f\phi)\,\mathrm{d}x\mathrm{d}y + \int_b [G\partial\phi/\partial n - (\phi - \phi_*)\partial G/\partial n]\,\mathrm{d}a, \tag{3.10}$$

where b is a fully-submerged closed surface. The classical integral identities (3.6a, b, c), on the other hand, take the same form for open sea-surface piercing surfaces and closed fully-submerged surfaces.

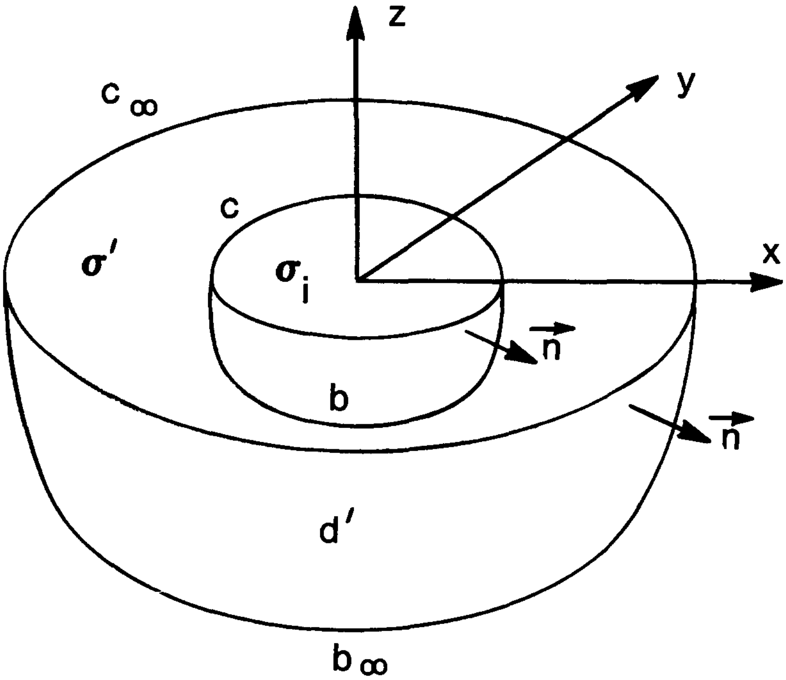

Figure 1. Definition sketch.

The integral identity (3.8) is valid for any point $\boldsymbol{\xi}$, whether outside, inside, or exactly on the body surface b. This new identity thus is essentially equivalent to the set of the three classical identities (3.6a, b, c), which are valid exclusively for $\boldsymbol{\xi}$ outside, inside, and on the body surface b, respectively. As a matter of fact, these three identities can be obtained from the identity (3.8) by noting that we have $fw_* + \int_b \partial G/\partial n \, da = 0$, 1, or 1/2 for $\boldsymbol{\xi}$ outside, inside, or on the body surface b, respectively, as may be verified from equations (3.7a), (3.7), (2.4) and (2.5). The classical identities (3.6a, b, c) likewise can be obtained from identity (3.10) in the case of a closed fully-submerged body by noting that we then have $\int_b \partial G/\partial n \, da = 0$, 1 or 1/2 for $\boldsymbol{\xi}$ outside, inside, or on the body surface b, respectively.

4. Particular cases

In the particular case when there is no body b, the integral identities (3.6b) and (3.6c) evidently are meaningless, and identities (3.6a), (3.8), and (3.9) all take the form

$$\phi_* = \int_{z<0} G\nabla^2\phi \, dv - \int_{z=0} G(\partial\phi/\partial z - f\phi) \, dx dy.$$

The above equation provides an explicit solution to the problem of determining the velocity potential $\phi(\boldsymbol{\xi})$ in the lower-half space $\zeta \leqslant 0$ when $\nabla^2\phi$ and $\partial\phi/\partial z - f\phi$ are specified in the lower-half space $z < 0$ and on the mean sea-surface $z = 0$, respectively. A classical problem of this type is that of determining the potential induced by a given pressure distribution $p(x,y)$ and/or a flux distribution $q(x, y)$ at the sea surface. The solution of this problem, stated in differential form by equations (2.1) and (2.3), is given by

$$\phi(\boldsymbol{\xi}) = -i\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}x\, G(\boldsymbol{\xi},x,y,z=0)\,[fp(x,y)+iq(x,y)]\,.$$

This solution is well known, and can be obtained by using more direct approaches than that adopted here. For instance, equations (2.1) and (2.3) can be solved directly by means of a double Fourier transformation with respect to the horizontal coordinates x and y. Another classical approach for determining the potential induced by a sea-surface pressure distribution is based on the use of a special Green function, $\bar{G}$ say, corresponding to the potential induced by a pressure impulse $p(x,y) = \delta(x-\xi)\,\delta(y-\eta)$ at the sea surface. This special Green function in fact is a particular case of the more general Green function used in this study; specifically, we have $\bar{G}(\boldsymbol{\xi},x,y) = -ifG(\boldsymbol{\xi},x,y,z=0)$.

Another important special problem is that of flow about a body in regular water waves, for which we have $\nabla^2\phi = 0$ in the mean flow domain d and $\partial\phi/\partial z - f\phi = 0$ on the mean sea surface σ. Unlike for the problem of flow induced by a sea-surface pressure and flux distribution, the integral identities obtained in the previous section do not provide an explicit solution to the problem of wave radiation and diffraction by a body. However, identities (3.6c) and (3.8) provide integral equations for determining the potential $\phi_* \equiv \phi(\boldsymbol{\xi})$ on the body surface b. These integral equations take the form

$$\phi_*/2 = \int_b G\partial\phi/\partial n\,\mathrm{d}a - \int_b \phi\partial G/\partial n\,\mathrm{d}a, \tag{4.1}$$

$$(1-fw_*)\phi_* = \int_b G\partial\phi/\partial n\,\mathrm{d}a - \int_b (\phi-\phi_*)\partial G/\partial n\,\mathrm{d}a, \tag{4.2}$$

corresponding to identities (3.6c) and (3.8), respectively. An alternative form, corresponding to identity (3.9), of the integral equation (4.2) is

$$\phi_* = \int_b \tilde{G}\partial\phi/\partial n\,\mathrm{d}a - \int_b (\phi-\phi_*)\,\partial\tilde{G}/\partial n\,\mathrm{d}a. \tag{4.2a}$$

In equations (4.2) and (4.2a), $w_* \equiv w(\boldsymbol{\xi})$ and $\tilde{G} \equiv \tilde{G}(\boldsymbol{\xi}, \mathbf{x})$ are the 'waterplane integral' and the 'modified Green function' defined by equations (3.8a) and (3.9a), respectively.

As was noted in the previous section equation (4.1) holds for $\boldsymbol{\xi}$ strictly on the body surface b, whereas equations (4.2) and (4.2a) are also valid in the mean flow domain d outside the body. This means that the latter equations can in principle be used to determine the potential ϕ in the entire solution domain $d+b$, for instance by using an iterative procedure based on a recurrence relation such as that given below by equation (4.5). However, it would usually be much simpler in practice to solve for ϕ on the body surface b, and – in the event (rare in reality) that knowledge of ϕ outside b is in fact required – to determine ϕ outside b by means of equation (3.6a), which here takes the simplified form

$$\phi_* = \int_b G\partial\phi/\partial n \, da - \int_b \phi\partial G/\partial n \, da. \tag{4.3}$$

Equations (4.2) and (4.2a) also hold for $\boldsymbol{\xi}$ inside the body, as was noted in the previous section. It thus might appear that these equations must also define the potential ϕ inside b. This result, were it true, would certainly be quite surprising, indeed fundamentally unacceptable, for it would mean that the 'exterior boundary-value problem' stated by equations (2.1), (2.2), and (2.3) would somehow define a potential inside the body. It can easily be shown, however, that equations (4.2) and (4.2a) allow the potential ϕ to be extended inside b in an entirely arbitrary manner. Indeed, equation (4.2) can be written in the form

$$C\phi_* = \int_b G\partial\phi/\partial n \, da - \int_b \phi\partial G/\partial n \, da, \tag{4.4}$$

where C is given by $C = 1 - fw_* - \int_b \partial G/\partial n \, da$. We have $C \equiv 0$ for $\boldsymbol{\xi}$ strictly inside b, as may be seen from equations (3.7a), (3.7), (2.4), and (2.5), and indeed was already shown in the previous section. Equation (4.4) therefore does not define ϕ_* inside b. For $\boldsymbol{\xi}$ outside b, on the other hand, we have $C \equiv 1$, and equation (4.4) becomes equation (4.3), which clearly defines ϕ_*.

An interesting feature distinguishing the new integral equation (4.2) from the classical integral equation (4.1) is that whereas the dipole integral in the usual integral equation (4.1) is discontinuous at the body surface b (as was noted in the previous section, and indeed is well known), the corresponding integral in equation (4.2) is a continuous function (the dipole density $\phi - \phi_*$ vanishes as the 'integration point' $\mathbf{x}$ and the 'field point' $\boldsymbol{\xi}$ coincide). As a matter of fact, the term $\phi_*/2$ on the left side of the integral equation (4.1) is correct only for points $\boldsymbol{\xi}$ where the body surface is smooth (that is, has a tangent plane), as was also noted previously. The classical integral equation (4.1) thus requires evaluation of a discontinuous function exactly on the surface of discontinuity of that function. This awkward problem, notably from the point of view of numerical calculations, is avoided in the integral equation (4.2). Furthermore, the integrand $(\phi - \phi_*)\partial G/\partial n$ in the dipole integral in equation (4.2) can be shown to be non-singular, that is to remain finite, as $\mathbf{x} \to \boldsymbol{\xi}$.

A choice of methods is available for solving the nonhomogeneous integral equation (4.2). In particular, a classical method of solution consists in using an iterative procedure. An obvious recurrence relation is that obtained by simply replacing ϕ by $\phi^{(k)}$ and $\phi^{(k+1)}$ on the right and left sides, respectively, of equation (4.2). This yields

$$(1 - fw_*)\phi_*^{(k+1)} = \psi_* - \int_b (\phi^{(k)} - \phi_*^{(k)})\partial G/\partial n \, da, \qquad k \geqslant 0, \tag{4.5}$$

where $\psi_* \equiv \psi(\boldsymbol{\xi})$ is the known (since $\partial\phi/\partial n$ is prescribed on b) nonhomogeneous term in equation (4.2) given by

$$\psi_* = \int_b G\partial\phi/\partial n\, da, \tag{4.6}$$

and the initial (zeroth) approximation $\phi^{(0)}$ must be specified somehow; for instance, one may simply take $\phi^{(0)} \equiv 0$.

An interesting modified form of the integral equation (4.2) may be obtained by expressing the potential in the form $\phi = k\psi$, where ψ is the potential given by equation (4.6) and k is the function defined as $k \equiv \phi/\psi$, in the manner used previously in [12]. By expressing the term $\phi - \phi_* \equiv \phi(\mathbf{x}) - \phi(\boldsymbol{\xi}) \equiv k(\mathbf{x})\psi(\mathbf{x}) - k(\boldsymbol{\xi})\psi(\boldsymbol{\xi}) \equiv k\psi - k_*\psi_*$ in equation (4.2) in the form $k_*(\psi - \psi_*) + (k - k_*)\psi$, and multiplying the thus-modified integral equation by ψ_*, we may obtain

$$\left[(1 - fw_*)\psi_* + \int_b (\psi - \psi_*)\partial G/\partial n\, da\right]\phi_* = \psi_*^2 - \int_b (\phi\psi_* - \psi\phi_*)\partial G/\partial n\, da. \tag{4.7}$$

If the potentials ϕ and ψ were proportional to one another, the term $\phi\psi_* - \psi\phi_*$ would vanish, and the modified integral equation (4.7) would yield the explicit solution

$$\phi_* = \psi_*^2 \Big/ \left[(1 - fw_*)\psi_* + \int_b (\psi - \psi_*)\partial G/\partial n\, da\right]. \tag{4.8}$$

This solution may be expected to provide a useful explicit approximation if the potentials ϕ and ψ are roughly proportional, that is if the function $k \equiv \phi/\psi$ is slowly varying. It is shown in [12] that for the special case of an open sea-surface piercing body in the shape of a half ellipsoid the explicit approximation (4.8) actually is exact for zero-frequency surge and sway motions and for infinite-frequency heave. Numerical calculations based on the integral equation (4.2) and the related explicit approximation (4.8) will be presented elsewhere.

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